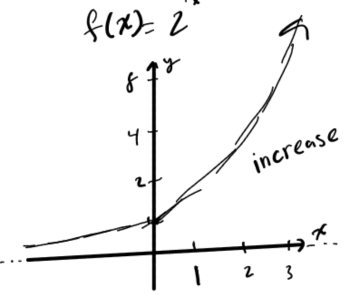
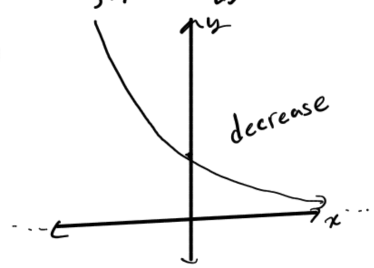
If you recall from algebra, this is how we defined exponentials and logarithms:

$f(x) = b^x$ exponential functions: $b > 0$, $b \neq 1$, $x \in \mathbb{R}$

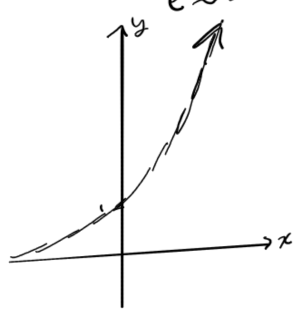
for example $f(x) = 2^x$



$f(x) = \left(\frac{1}{2}\right)^x$



② $f(x) = e^x$ "Euler's Number" $e \approx 2.7$



③ $f(x) = \log_b x$, $b > 0$, $b \neq 1$, inverse b^x

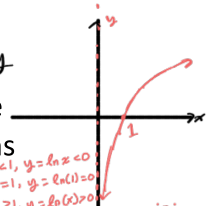
$y = \log_b x \iff x = b^y$

output, base, input, switch

$\log_2 8 = b$, $8 = 2^y$, $y = 3$, $\log_2 8 = 3$

④ $f(x) = \ln(x)$ natural log function $\log_e(x)$

$y = \ln(x) \iff x = e^y$



Now we are going to *define* a new way of looking at logarithms, which can smooth out some of the difficult technical problems we encountered before. It turns out that this way of defining logarithms and exponentials give us similar properties that we were familiar with, so it all works out nicely!

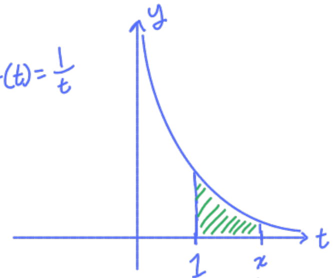
Definition: The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad \boxed{x > 0}$$

t is a "dummy variable"

Geometric interpretation

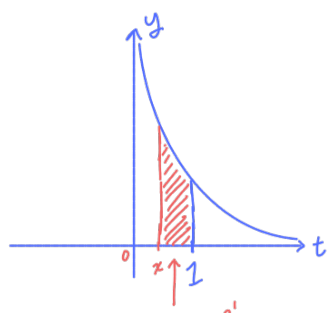
Case 1: $x > 1$



$\ln(x) = \int_1^x \frac{1}{t}\,dt = \text{Area} > 0$

$\boxed{x > 1 \Rightarrow \ln(x) > 0}$

Case 2: $0 < x < 1$



$\text{Area} = \int_x^1 \frac{1}{t}\,dt > 0$

$\ln(x) = \int_1^x \frac{1}{t}\,dt = \boxed{-\int_x^1 \frac{1}{t}\,dt} < 0$

If $0 < x < 1 \Rightarrow \ln x < 0$

Case 3: $x = 1$

$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

$\boxed{\ln(1) = 0}$

Conclusion:

If $x > 1$, then $\ln x = \int_1^x \frac{1}{t}\,dt > 0$

If $x = 1$, then $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

If $0 < x < 1$, then $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$

The beauty of such a definition is that this integral looks like the form of integrals we see in FTC I.

differential operator

$\downarrow$

$\frac{d}{dx}(\quad)$

$\uparrow$

function

$\frac{d}{dx}(x^2)=2x$

If we take the derivative, we get:

FTC I: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^{x}\frac{1}{t}\,dt = \frac{1}{x}$$

$f'(u(x))\cdot u'(x)$

So the **derivative** of the natural log function

$$\frac{d}{dx}(\ln x) = \frac{1}{x}\quad,\quad x>0$$

(Chain Rule: $\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}$)

u'

$\ln\square \xrightarrow{\frac{d}{dx}} \frac{\square'}{\square}$

Example 1: Practice with Differentiation

Find $\frac{dy}{dx}$

a) $y=\ln\left(3x^2+5\right)$

$$\frac{dx}{dx} = \frac{d}{dx}\ln(\overbrace{3x^2+5}^{u})$$ Chain Rule

$$= \frac{1}{3x^2+5}\cdot(3x^2+5)'$$

$$= \frac{6x}{3x^2+5}$$

b) $y=\ln\left(\sin^2 x\right) = \ln[(\sin x)^2]$

$$\frac{dy}{dx} = \frac{1}{\sin^2 x}\cdot\frac{d}{dx}(\sin x)^2$$

$$= \frac{1}{\sin^2 x}\cdot 2\sin x\cdot\frac{d}{dx}(\sin x)$$

$$= \frac{1}{\sin^{\cancel{2}} x}\cdot 2\cancel{\sin x}\cdot\cos x$$

$$= \frac{2\cos x}{\sin x}$$

$$= \boxed{2\cot x}$$

$$y = u \cdot v$$
$$y' = u'v + v'u$$
$$= u \cdot v' + u'v$$
$$= u'v + uv'$$

Find $\frac{dy}{dx}$

c) $y = (\cos x)(\ln \tan x)^2$

Leibnitz

$$y' = \frac{dy}{dx} = \left[\frac{d}{dx}(\cos x)\right](\ln \tan x)^2 + \cos x\left[\frac{d}{dx}(\ln \tan x)^2\right]$$
$$= (-\sin x)(\ln \tan x)^2 + \cos x\left[2(\ln \tan x)\cdot\frac{1}{\tan x}\cdot \sec^2 x\right]$$
$$= (-\sin x)(\ln \tan x)^2 + \frac{2(\ln \tan x)\cos x \sec^2 x}{\tan x}$$
$$= \boxed{(-\sin x)(\ln \tan x)^2 + \frac{2(\ln \tan x)\sec x}{\tan x}}$$

d) $y = \frac{\ln x}{\sqrt{1-x}}$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$y = \frac{u}{v}$$
$$y' = \frac{u'v - v'u}{v^2}$$

$$\frac{f}{g}$$
$$\frac{f'g - fg'}{g^2}$$

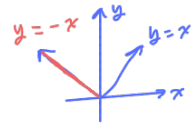
$$\frac{dy}{dx} = \frac{\left[\frac{d}{dx}(\ln x)\right]\sqrt{1-x} - \ln x \cdot \frac{d}{dx}\sqrt{1-x}}{(\sqrt{1-x})^2}$$
$$= \frac{\frac{1}{x}\cdot\sqrt{1-x} - \ln x \cdot \frac{1}{2\sqrt{1-x}}\cdot(-1)}{1-x}$$
$$= \frac{\left(\frac{\sqrt{1-x}}{\cancel{x}} + \frac{\ln x}{\cancel{2\sqrt{1-x}}}\right)\cdot \cancel{2}x\,\cancel{\sqrt{1-x}}}{(1-x)^1\, 2x\sqrt{1-x}}$$

$$\rightarrow (1-x)^1(1-x)^{\frac{1}{2}} = (1-x)^{1+\frac{1}{2}}$$

$$= \boxed{\frac{2(1-x) + x\ln x}{2x(1-x)^{3/2}}}$$

Example 2: Find the derivative of $\boxed{f(x)=\ln|x| \;,\; x\neq 0}$

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



$|-3| = -(-3) = 3$, $x=-3$

$f(x) = \ln|x| = \begin{cases} \ln(x) & \text{if } x>0 \\ \ln(-x) & \text{if } x<0 \end{cases}$

$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x>0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x<0 \end{cases}$

Template

Therefore: $\quad \dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$ for $x \neq 0$ and $\boxed{\int \dfrac{1}{x}\,dx = \ln|x| + c}$, $x \neq 0$ and $\boxed{\int \dfrac{1}{u}\,du = \int \dfrac{du}{u} = \ln|u| + c}$

Example 3: Evaluate the integrals. These require using a substitution!

Goal: Get the integrals to look like $\int \frac{1}{u}\,du$ or $\int \frac{du}{u}$ (The "template" for getting an $\ln$ antiderivative).

(a) $\displaystyle\int \frac{dx}{x\ln x}$

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$\cancel{x\,du = dx}$

$\int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$

$= \int \frac{1}{u}\,du = \int \frac{du}{u}$

$= \ln|u| + c$

$= \ln|\ln x| + c$

(b) $\displaystyle\int \frac{\cos x}{2+3\sin x}\,dx$

Let $u = 2 + 3\sin x$

$\frac{du}{dx} = 3\cdot\cos x$

$du = 3\cdot\cos x\,dx$

$= \int \frac{1}{3u}\,du$

$= \frac{1}{3}\int \frac{du}{u}$

$= \frac{1}{3}(\ln|u| + c)$

$= \frac{1}{3}\ln|2+3\sin x| + c$

keep absolute value bars because $2+3\sin x$ can be negative!

$= \boxed{\frac{1}{3}\ln|2+3\sin x| + c}$

(c) Evaluate $\displaystyle\int_0^2 \frac{x^2}{4x^3+1}\,dx$

Let $u = 4x^3+1$

$4(2)^3+1 = 33$

$4(0)^3+1 = 1$

$\frac{du}{dx} = 12x^2$

$\frac{du}{12} = x^2\,dx$

$= \int_1^{33} \frac{1}{u}\cdot\frac{1}{12}\,du$

$= \frac{1}{12}\int_1^{33} \frac{du}{u}$

$= \frac{1}{12}\Big[\ln|u|\Big]_1^{33}$

$= \frac{1}{12}(\ln(33) - \ln(1)) = \boxed{\frac{1}{12}(\ln 33)}$

Remember!!

Always adjust you upper and lower bounds because by default, they are in terms of x, not u.

$\int \frac{du}{u}$

(d) Evaluate $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

Let $u = \cos x$

$du = -\sin x \, dx$

$\sin x \, dx = -du$

$= \int \frac{-du}{u}$

$= -\ln|u| + c$

$\int \tan x \, dx = -\ln|\cos x| + c$

$\ln|\cos x|^{-1} + c$

$= \boxed{\ln|\sec x| + c}$

$\ln(x^r)$

$= r \ln x$

(e) Evaluate $\int \sec x \, dx$

$= \int \frac{\sec x}{1} \cdot \frac{(\sec x + \tan x)}{\sec x + \tan x} dx$

$= \int \frac{\overbrace{\sec^2 x + \sec x \tan x \, dx}^{du}}{\sec x + \tan x}$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) dx$

$= (\sec^2 x + \sec x \tan x) dx$

$= \int \frac{du}{u}$

$= \ln|u| + c$

$= \ln|\sec x + \tan x| + c$

Make sure to **try for yourself** to prove the formulas for $\int \cot x \, dx$ and $\int \csc x \, dx$

Summary: (These should be memorized but you should also know how to prove them.)

$\int \tan x \, dx = \ln|\sec x| + c$

$\int \cot x \, dx = \ln|\sin x| + c$

$\int \sec x \, dx = \ln|\sec x + \tan x| + c$

$\int \csc x \, dx = \ln|\csc x - \cot x| + c$

Laws of Logarithms

If x and y are positive numbers and r is a rational number, then

goal

1. $\ln(xy) = \ln x + \ln y$ (Product Rule)
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ (Quotient Rule)
3. $\ln(x^r) = r\ln x$ (Power Rule)

Make sure to **try for yourself** to prove the Quotient and Power Rules!

<u>Proof of the Product Rule</u> $\boxed{\ln(xy) = \ln x + \ln y:}$ goal

making y a constant

Let $f(x) = \ln(ax)$ where a is a positive constant, and let $g(x) = \ln x$.

Chain rule

Then $f'(x) = \frac{\cancel{a}}{\cancel{a}x} = \frac{1}{x}$ and $g'(x) = \frac{1}{x}$

So we see that $f(x) = \ln(ax)$ and $g(x) = \ln x$ have the same derivative. Thus they must come from the same family of functions and differ only by a constant. (If $f'(x) = g'(x)$ then $f(x) = g(x) + C$)

Therefore, we can write $\ln(ax) = \ln x + C$ ← $\ln(a)$

If we let $x = 1$, we get: $\ln(a) = \ln(1) + c$

$\ln(a) = 0 + c$

$\therefore c = \ln(a)$

$\ln(ax) = \ln x + \ln a$

y y

But a was arbitrary and can now be replaced by any number y, which leads to:

$\ln(yx) = \ln x + \ln y$

$\Rightarrow \ln(xy) = \ln x + \ln y$

Q.E.D. or ■

(Quod Erat Demonstrandom)

Warnings: Be careful with the correct application of these rules!

$\ln(2x) = \ln 2 + \ln x$

$\boxed{(\ln 2)(\ln x)} \neq \ln(2x)$

$\ln(x^3) = 3\ln x \neq [\ln x]^3$

$\ln(2+x) \neq \ln 2 + \ln x = \ln(2x)$

$\ln\left(\frac{3}{4}\right) = \ln 3 - \ln 4$

$\ln\left(\frac{3}{4}\right) \neq \cancel{\frac{\ln 3}{\ln 4}}$

Using logarithmic rules to simplify a function:

Example 4 Find the derivative of $y = \ln\left(\dfrac{5x}{\sqrt[3]{x-1}}\right)$

$$y = \ln(5x) - \ln(x-1)^{\frac{1}{3}}$$

$$y = \ln 5 + \ln x - \frac{1}{3}\ln(x-1)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln 5 + \ln x - \frac{1}{3}\ln(x-1)\right]$$

$$= \frac{1}{x} - \frac{1}{3}\cdot\frac{1}{x-1}$$

$$= \boxed{\frac{1}{x} - \frac{1}{3x-3}}$$

Technique of Logarithmic Differentiation

Example 5 Differentiate $y = \dfrac{\left(x^3+1\right)^4 \sin^2 x}{x^2\sqrt{2x+5}}$

Step 1) Apply ln to both sides of the equation, and use the law of logarithms to simplify (expand) the right side of the equation.

$$\ln y = \ln\left(\frac{(x^3+1)^4 \sin^2 x}{x^2(2x+5)^{1/2}}\right)$$

$$= \ln\left((x^3+1)^4 \sin^2 x\right) - \left[\ln\left(x^2(2x+5)^{1/2}\right)\right]$$

$$= \ln(x^3+1)^4 + \ln(\sin^2 x) - \ln(x^2) - \ln(2x+5)^{1/2}$$

$$\ln y = 4\ln(x^3+1) + 2\ln(\sin x) - 2\ln(x) - \tfrac{1}{2}\ln(2x+5)$$

Step 2) Differentiate implicitly with respect to the independent variable (in this case x).

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left[4\ln(x^3+1) + 2\ln(\sin x) - 2\ln(x) - \tfrac{1}{2}\ln(2x+5)\right]$$

$$\frac{y}{1}\cdot\frac{1}{y}\cdot\frac{dy}{dx} = \left[4\cdot\frac{3x^2}{x^3+1} + 2\cdot\frac{\cos x}{\sin x} - 2\cdot\frac{1}{x} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{2x+5}\right]\cdot\frac{y}{1}$$

Step 3) Solve the resulting equation for y' or $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{(x^3+1)^4\sin^2 x}{x^2\sqrt{2x+5}}\left(\frac{12x^2}{x^3+1} + 2\cot x - \frac{2}{x} - \frac{1}{2x+5}\right)$$

(**TRY FOR YOURSELF**) Example 6) Differentiate $y = \sqrt{\dfrac{x+1}{2x-5}}$

Graph and properties of $y = \ln x$

$x > 0$ (no y-intercept)

$\ln(1) = 0 \leftarrow \int_1^1 \frac{1}{t}\,dt$ x-intercept $(1, 0)$

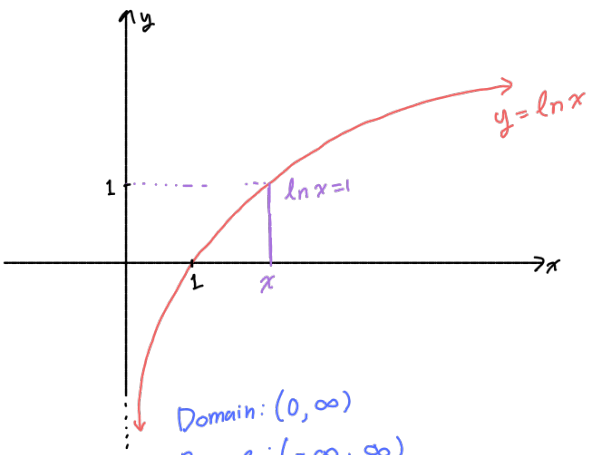
$y = \ln x, \quad x > 0$

$\frac{dy}{dx} = \frac{1}{x}, \quad x > 0 \qquad \frac{dy}{dx} = x^{-1}$

graph is increasing always

$\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ $- -$ $+ +$

graph is concave down



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

$\boxed{\text{Limit Function}}$: $\lim_{x \to 0^+} \ln x = -\infty$

$\lim_{x \to \infty} \ln x = \infty$

Since $\ln 1 = 0$ and $\ln x$ is an increasing continuous function, the Intermediate Value (I. V. T.) theorem says that there is a number x in the interval $(1, \infty)$ such that $f(x) = 1$, i.e. $\ln x = 1$. This number is denoted as e.

Definition: e is the number such that $\ln e = 1$.

Approximate value of e

Euler's number (Irrational Number)

Andrew Jackson 7th president, in 1828, 2 terms

$e \approx 2.7\ 1828\ 1828\ 45\ 90\ 45\ 235\ 360$

First 3 Prime numbers

degrees in a circle

more digits in easter egg link in education resources page