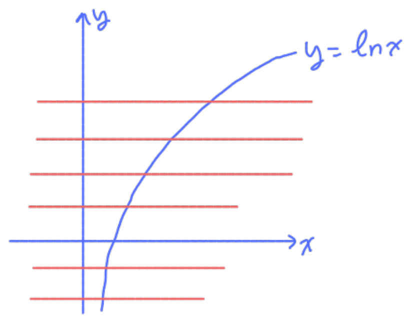
Math 8 Handout (Stewart) **<u>Section 6.3*</u>: The Natural Exponential Function** **I. Lai**

<u>Definition of the e^x function</u>

$f(x) = \ln x$ is a one-to-one function because



• the graph passes the horizontal line test.

• $f'(x) = \frac{1}{x} > 0$ since $x > 0$

f is always increasing

$\therefore$ f is one-to-one.

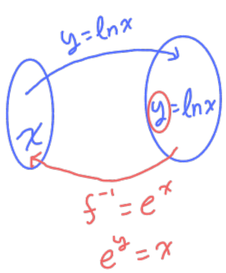
(therefore)

Since $f(x) = \ln x$ is a one-to-one function, it is <u>invertible</u> and has an ___inverse function___.

We will define this inverse function to be $f^{-1}(x) = \boxed{e^x}$ where e is Euler's constant.

as the inverse of $\ln x$

By definition, we have the inverse relationship: If f and f^{-1} are inverse functions, then



$$f(x) = y \Leftrightarrow x = f^{-1}(y)$$

$\underset{\uparrow \text{ input}}{\ln(x)} = \underset{\uparrow \text{ output}}{y} \iff \underset{\uparrow \text{ output}}{x} = e^{y}$ ↙ input

$3 = \ln_e x$

$e^3 = x$

Examples: $3 = \ln x \Rightarrow x = e^3$

So $\ln x = y \Leftrightarrow x = \boxed{e^y}$ and $e^x = y \Leftrightarrow x = \boxed{\ln(y)}$

<u>Inverse Function Cancellation properties:</u>

$f^{-1}(f(x)) = x$ for x in the domain of f and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}

inner

outer → $e^{\ln x} = x$ for $x > 0$ and $\ln e^x = x$ for $x \in \mathbb{R}$

$f^{-1}(f(x))$ if x is in the domain of $\ln(x)$

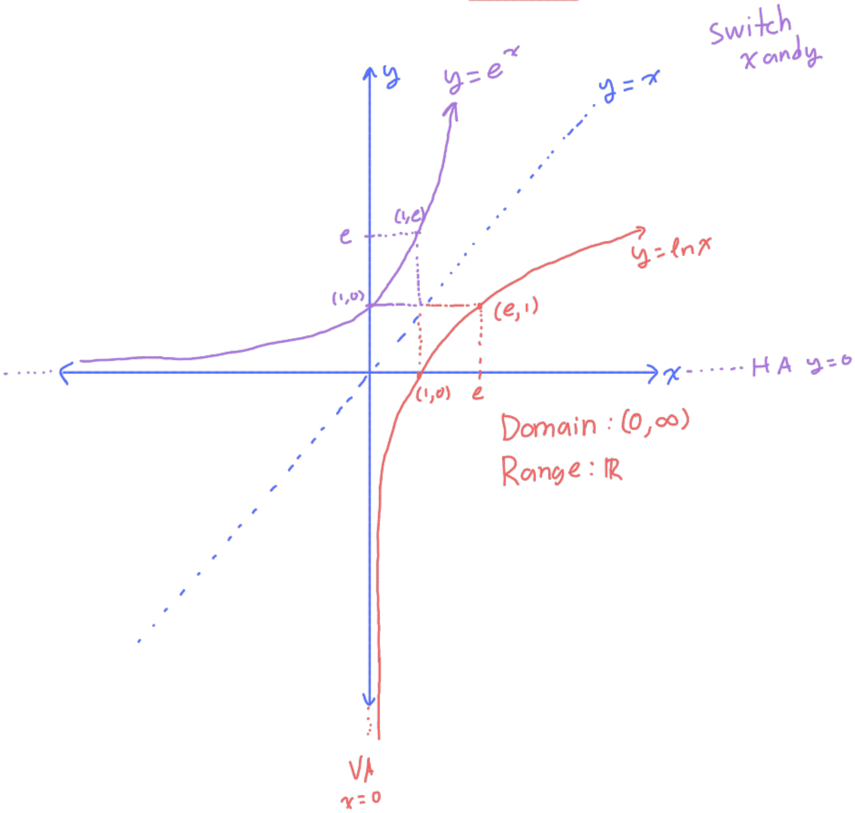
domain of $e^x = \mathbb{R}$

Examples: $\ln e^3 = 3$

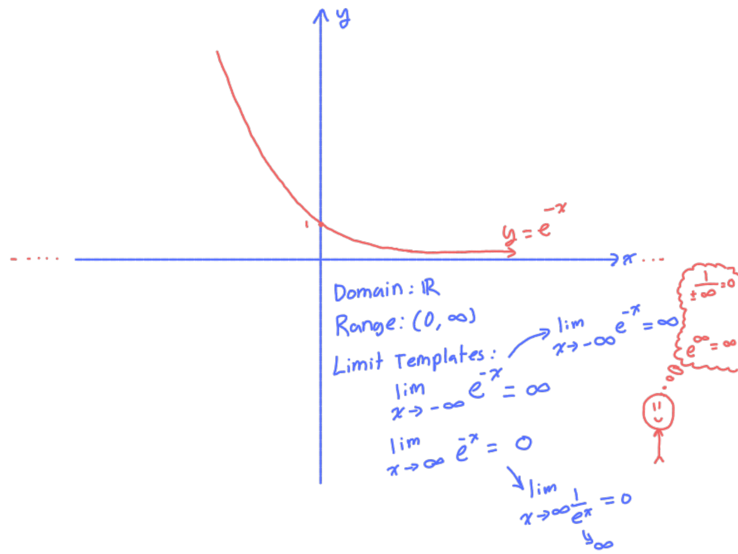
$e^{\ln 10} = 10$

Graphs and properties of $y=e^x$ **and** $y=e^{-x}$

purple $\boxed{y=e^x}$ (use the graph of red $\boxed{y=\ln x}$ as a guide)



$y=e^{-x}$ (use a transformation from $y=e^x$)



Domain : $\mathbb{R}$

Range : $(0,\infty)$ $\Rightarrow e^x>0$

$$\left.\begin{array}{l}\lim_{x\to-\infty} e^x=0\\ \lim_{x\to\infty} e^x=\infty\end{array}\right\}\text{ limit templates for } e^x$$

Limits

Evaluate the following limits.

1) $\lim_{x\to 2^-} e^{\frac{3}{2-x}}$

using limit templates

$= \lim_{u\to\infty} e^u$

$= \boxed{\infty}$

~~$e^{\frac{3}{2-2^-}} = e^{\frac{3}{0^+}} = e^{\infty} = \infty$~~

let $u = \frac{3}{2-x}$

As $x \to 2^-$ $\Rightarrow x < 2$

$u = \frac{3}{2-x} \to$

side work

$\frac{3}{\text{very small positive number } (0^+)} \to \infty$

$u \to \infty$

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2) $\lim_{x\to\infty} \frac{e^{3x}}{e^{3x}+1}$

$\lim_{x\to\infty} \frac{(3x^2+5x)/x^2}{(6x^2-4)/x^2} = \lim_{x\to\infty} \frac{3+\cancel{5/x}^{\,0}}{6-\cancel{4/x^2}^{\,0}}$

$= \frac{3}{6} = \frac{1}{2}$

(This limit gives an <u>indeterminate form</u> of $\boxed{\frac{\infty}{\infty}}$. How do we deal with such forms?)

$\lim_{x\to\infty} \frac{(e^{3x})/e^{3x}}{(e^{3x}+1)/e^{3x}}$

$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{e^{3x}}}$

$= \lim_{x\to\infty} \frac{1}{1+\cancel{e^{-3x}}^{\,0}}$

$= \frac{1}{1}$

$= 1$

$\lim_{x\to\infty} e^{-x} = 0$

$\lim_{x\to\infty} e^{-3x} = 0$

Laws of Exponents

If x and y are real numbers and r is a rational number,

(1) $\boxed{e^x e^y = e^{x+y}}$ (Product Law)

(2) $\frac{e^x}{e^y} = e^{x-y}$ (Quotient Law)

(3) $\left(e^x\right)^r = e^{rx}$ (Power Law)

Apply ln to the left side of the equation.

Proof of (1) Product Law (**try-for-yourself:** make sure you can prove the other laws too)

Given: x and y are real numbers (we want to try to prove $e^x e^y = e^{x+y}$)

Brainstorm: use ln
product rule
for ln

Using the Product Law for $\ln$, we can write

$$\ln\left(e^x e^y\right) = \ln e^x + \ln e^y$$ product rule for ln

$$= x + y$$ cancellation properties of inverse functions

$$\underbrace{\ln(e^x e^y)} = \ln\underbrace{(e^{x+y})}$$

Since ln is a one-to-one function,

$$e^x e^y = e^{x+y}$$

Q.E.D.

CALCULUS of e^x

Derivative of e^x $$\frac{d}{dx}\left(e^x\right)=e^x$$

Proof:

6.1 Since $f(x)=\ln x$ is differentiable and the inverse of a differentiable function is also differentiable, therefore $f(x)=e^x$ is differentiable.

Let $y=e^x$ (Our goal is to find $\frac{dy}{dx}$)

Then by definition of inverse functions: $\ln y = \boxed{x}$

Using implicit differentiation, we get: $\frac{d}{dx}\ln y = \frac{d}{dx}(x)$

$$\frac{y}{1}\cdot\frac{dy}{dx}\cdot\frac{1}{y} = 1\cdot\frac{y}{1}$$

$$\frac{dy}{dx} = y = e^x$$

$$\therefore \frac{dy}{dx} = e^x$$

Q.E.D.

Chain Rule $\frac{d}{dx}\left(e^u\right)=e^u\cdot\frac{du}{dx}$

In particular: $\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$ for a constant a

For example, $\frac{d}{dx}\left(e^{-2x}\right)=-2e^{-2x}$

Example 3 Differentiate each function with respect to its independent variable.

$e^{u} \cdot u'$

(a) $f(x) = e^{x\ln x}$

chain rule

$f'(x) = e^{x\ln x} \cdot \frac{d}{dx}(x\ln x)$

$= e^{x\ln x} \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right)$

$= e^{x\ln x} \cdot (1 + \ln x)$

$= e^{x\ln(x)} \cdot \ln(x) e^{x\ln(x)}$

(b) $g(t) = \sec^3\left(e^t\right)$

$\frac{d}{dt}\sec^3(e^t) = 3\sec^2(e^t) \cdot \frac{d}{dt}\left[\sec(e^t)\right]$

$= 3\sec^2(e^t)\sec(e^t)\tan(e^t) \cdot \frac{d}{dt}(e^t)$

$= 3\underline{\sec^2(e^t)}\,\underline{\sec(e^t)}\tan(e^t)e^t$

$= \boxed{3\sec^3(e^t)\tan(e^t)e^t}$

Integral of e^x $\boxed{\int e^x\,dx = e^x + C}$ **or** $\boxed{\int e^u\,du = e^u + C}$

In particular: $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$ for a constant a (by using a u-substitution)

For example: $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

$\int e^{5x}dx$

$= \frac{1}{5}\int e^u du$

$= \frac{1}{5}e^u + c$

$= \frac{1}{5}e^{5x} + c$

Let $u = 5x$

$du = 5dx$

$dx = \frac{1}{5}du$

$\int \frac{7}{e^{3x}}dx = \int 7e^{-3x}dx$

$= 7 \cdot \frac{1}{-3}e^{-3x} + c$

$= -\frac{7}{3}e^{-3x} + c$

Examples Evaluate each integral.

4) $\int e^{\boxed{\cot x}}\boxed{(\csc^2 x)\,dx}$

$\int e^u du = e^u + c$

Let $u = \cot x$

$du = -\csc^2 x\,dx$

$-du = \csc^2 x\,dx$

$= \int e^u(-du)$

$= -\int e^u du$

$= -e^u + c$

$= -e^{\cot x} + c$

5) $\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}dx$

Let $u = 1 + e^{-x}$ |1.1| Remember, $\sqrt{1+e^{-x}}$ is a composite function.

$du = -e^{-x}dx$

$-du = e^{-x}dx$

$= \frac{1}{e^x}dx$

LB(lower bound)

when $x = 0$

$u(0) = 1 + e^0 = 2$

UB (upper bound)

when $x = 1$

$u(1) = 1 + e^{-1} = 1 + \frac{1}{e}$

$= \int_2^{1+\frac{1}{e}} \sqrt{u}(-du)$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$= \int_{1+\frac{1}{e}}^2 \sqrt{u}\,du$

$= \frac{2}{3}\left[u^{3/2}\right]_{1+\frac{1}{e}}^2$

$= \frac{2}{3}\left(2^{3/2} - \left(1+\frac{1}{e}\right)^{3/2}\right)$