Math 8 Handout (Stewart) **<u>Section 6.4*</u>: General Logarithmic and Exponential Functions** **I. Lai**

For $b > 0$ and r a rational number, we can use known properties to write the following:

$$\underline{b^r} = e^{\ln b^r} = \underline{e^{r\ln b}}$$

cancellation property

power law for ln

$\ln x$

e

property of inverse functions

So therefore, $\boxed{b^r = e^{r\ln b}}$. This is true for all rational r.

We will extend this idea to all real numbers, and use this relationship to define b^x.

Definition of b^x

$$\boxed{b^x = e^{x\ln b}} \qquad \boxed{\underline{b>0}\ ,\ x \in \mathbb{R}}$$

because → $(-4)^x$ where x is a fraction, you could get a negative root, which is not real.

Recall the expression $2^{\sqrt{3}}$? Now, we can use this definition to write $2^{\sqrt{3}}$.

$$b^x = \underline{e^{x\ln b}}$$

$$2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \qquad 5^3 = e^{3\ln 5}$$

$b=2$ $x=\sqrt{3}$

The function $f(x) = b^x$ is called the **exponential function with base** b.

Additionally, with this definition, the power rule for the ln function ($\ln b^r = r\ln b$ for rational r) can now be extended to be true for all real numbers.

$$\ln b^x = \ln e^{x\ln b}$$
$$= x\ln b$$

Therefore, $\ln b^x = x\ln b$ for any real number x.

Laws of Exponents

If x and y are real numbers and $a, b > 0$, then

1. $\underline{b^x b^y = b^{x+y}}$ (Product Law)
2. $\dfrac{b^x}{b^y} = b^{x-y}$ (Quotient Law)
3. $(b^x)^y = b^{xy}$ (Power Law)
4. $\underbrace{(ab)^x = a^x b^x}$ (Product-to-power Law)

$2^3 \cdot 2^5 = 2^8$ $\qquad$ $2^\pi \cdot 2^{\sqrt{3}} = 2^{\pi+\sqrt{3}}$

e properties ↓

$b^x = e^{x\ln b}$

$(2x)^3 = 2^3 \cdot x^3$

$= 8x^3$

Proof of (1) Product Law: (**try-for-yourself**: prove the other three laws)

Given: $b > 0$ and $x, y \in \mathbb{R}$ Goal: To prove: $b^x b^y = b^{x+y}$

$LHS = b^x b^y$

$= e^{x \ln b} \cdot e^{y \ln b}$ by definition of b^v

$= e^{x \ln b + y \ln b}$ $e^v \cdot e^v = e^{v+v}$ product Law for e

$= e^{(x+y) \ln b}$ factor $\ln b$

$= b^{x+y} = RHS$ $e^{\square \ln b} = b^{\square}$ by definition

$b^x b^y = b^{x+y}$ QED

Derivative of the exponential function with base b:

$$\boxed{\frac{d}{dx}\left(b^x\right) = b^x \ln b}$$

Chain Rule: $\frac{d}{dx} b^u = b^u \ln b \left(\frac{du}{dx}\right)$

Example: $\frac{d}{dx}\left(5^x\right) = 5^x \ln 5$ $\frac{d}{dx}(7^x) = 7^x \ln(7)$ $\frac{d}{dx}(e^x) = e^x \underbrace{\ln e}_{1} = e^x$

Proof:

$$\frac{d}{dx}\left(b^x\right) = \frac{d}{dx}\left(e^{x \ln b}\right)$$ definition of $b^x = e^{x \ln x b}$ $\frac{d}{dx} e^v = e^v \cdot v'$

$$= e^{x \ln b} \frac{d}{dx}\left(x \ln b\right)$$

$$= e^{x \ln b} \underbrace{\frac{d}{dx}\left(\underbrace{\left(\ln b\right)}_{\text{constant}} x\right)}$$ $\frac{d}{dx}(kx) = \underset{\ln b}{k}$

$$= \underbrace{e^{x \ln b}} \ln b$$

$$= b^x \ln b$$

QED

Calc I. power rule

$$\frac{d}{dx}(x^{n}) = nx^{n-1}$$

constant exponent

variable in the base

$$\frac{d}{dx}(e^{x}) = e^{x}$$

variable exponent

constant base

$$\frac{d}{dx}(b^{x}) = b^{x}\ln b$$

variable exponent

b>0 constant here

Example 1: Find the derivative.

(a) $y=\left(3^{2-x^2}\right)\left(x^{\sqrt{2}}\right)$

$$\frac{dy}{dx} = \left[\frac{d}{dx}\left(3^{2-x^2}\right)\right]x^{\sqrt{2}} + 3^{2-x^2}\cdot\left[\frac{d}{dx}\left(x^{\sqrt{2}}\right)\right]$$

$$= -3^{2-x^2}\ln(3)\,2x \quad x^{\sqrt{2}} + 3^{2-x^2}\quad \sqrt{2}\,x^{\sqrt{2}-1}$$

$$= 3^{2-x^2}\left[-2x\ln(3)\,x^{\sqrt{2}} + \sqrt{2}\,x^{\sqrt{2}-1}\right]$$

(b) $y=\underbrace{x^{e}}+\underbrace{e^{x}}+\underbrace{\sqrt{3}^{x}}+\underbrace{3^{\sqrt{2}}}_{\text{constant}}$

$$\frac{dy}{dx} = ex^{e-1} + e^{x} + \sqrt{3}^{x}\ln\sqrt{3} + 0$$

(c) $y=\left(e^{3x}+\ln x\right)^{\sqrt{\cos x}}$

Method One: Logarithmic Differentiation

$$\ln y = \ln\left(e^{3x}+\ln x\right)^{\sqrt{\cos x}}$$

$$\ln y = \sqrt{\cos x}\cdot\ln\left(e^{3x}+\ln x\right)$$

$$\frac{dy}{dx}\cdot\frac{1}{y} = \frac{-\sin x}{2\sqrt{\cos x}}\cdot\ln\left(e^{3x}+\ln x\right) + \sqrt{\cos x}\cdot\frac{x\left(3e^{3x}+\frac{1}{x}\right)}{x\left(e^{3x}+\ln x\right)}$$

$$= \frac{(-\sin x)\ln\left(e^{3x}+\ln x\right)}{2\sqrt{\cos x}} + \frac{\sqrt{\cos x}\left(3xe^{3x}+1\right)}{x\left(e^{3x}+\ln x\right)}$$

$$= \left(e^{3x}+\ln x\right)^{\sqrt{\cos x}}\left[\frac{(-\sin x)\ln\left(e^{3x}+\ln x\right)}{2\sqrt{\cos x}} + \frac{\sqrt{\cos x}\left(3xe^{3x}+1\right)}{x\left(e^{3x}+\ln x\right)}\right]$$

Method II: By definition $b^{u} = e^{u\ln b}$

$$\left(e^{3x}+\ln x\right)^{\sqrt{\cos x}} = e^{\sqrt{\cos x}\,\ln\left(e^{3x}+\ln x\right)}$$

$$\frac{d}{dx}\left(e^{3x}+\ln x\right)^{\sqrt{\cos x}} = \frac{d}{dx}\,e^{\sqrt{\cos x}\,\ln\left(e^{3x}+\ln x\right)}$$

$$= e^{\sqrt{\cos x}\,\ln\left(e^{3x}+\ln x\right)}\cdot\frac{d}{dx}\sqrt{\cos x}\,\ln\left(e^{3x}+\ln x\right)$$

$$\frac{dy}{dx} = e^{\sqrt{\cos x}\,\ln\left(e^{3x}+\ln x\right)}\left[\frac{-\sin x}{2\sqrt{\cos x}}\cdot\ln\left(e^{3x}+\ln x\right) + \sqrt{\cos x}\cdot\frac{3e^{3x}+\frac{1}{x}}{e^{3x}+\ln x}\right]$$

$$= \left(e^{3x}+\ln x\right)^{\sqrt{\cos x}}\left[\frac{(-\sin x)\ln\left(e^{3x}+\ln x\right)}{2\sqrt{\cos x}} + \frac{\sqrt{\cos x}\left(3xe^{3x}+1\right)}{x\left(e^{3x}+\ln x\right)}\right]$$

Summary For constants b, n

$$\frac{d}{dx}b^{n} = 0$$

Calc I power rule

constant

$$\frac{d}{dx}\left[f(x)\right]^{n} = n\left[f(x)\right]^{n-1}\,f'(x)$$

$$\frac{d}{dx}b^{g(x)} = b^{g(x)}\ln b\cdot g'(x)$$

$\frac{d}{dx}\left[f(x)\right]^{g(x)}$ → Use logarithmic differentiation.

OR rewrite b^{u} as $e^{u\ln b}$ and differentiate.

$$\frac{d}{dx}(b^x) = b^x \cdot \ln b$$

Integral of the exponential function with base b:

$$\underline{\int b^x\,dx = \frac{1}{\ln b}\left(b^x\right) + C} \quad ; \quad \int b^u\,du = \frac{1}{\ln b}\left(b^u\right) + C$$

Proof:

$$\int b^x\,dx = \int e^{x \ln b}\,dx$$

$$= \int e^u \cdot \frac{1}{\ln b}\,du$$

$$= \frac{1}{\ln b}\int e^u\,du$$

$$= \frac{1}{\ln b}\,e^u + c$$

$$= \frac{1}{\ln b} \cdot e^{x \ln b} + c$$

$$= \frac{1}{\ln b} \cdot b^x + c$$

Let $u = x(\ln b)$

$$\frac{du}{\ln b} = \frac{\ln b\,dx}{\ln b}$$

$$\frac{1}{\ln b}\,du = dx$$

Examples:

$$\int 10^x\,dx = \frac{10^x}{\ln 10} + c$$

or

$$= \frac{1}{\ln 10}\left(10^x\right) + c$$

$$\int \left(\tfrac{1}{5}\right)^x dx$$

$$= \frac{\left(\frac{1}{5}\right)^x}{\underbrace{\ln\left(\frac{1}{5}\right)}_{\ln 5^{-1}}} + c$$

$-\ln 5$

$$= \frac{1}{-\ln 5}\left(\tfrac{1}{5}\right)^x + c$$

Example 2: Evaluate the integral.

$$\int \frac{10^{\sqrt{x}}}{\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$2\,du = \frac{1}{\sqrt{x}}\,dx$$

$$= 2\int 10^u\,du$$

$$= 2\left(\frac{10^u}{\ln 10}\right) + c$$

$$= 2\left(\frac{10^{\sqrt{x}}}{\ln 10}\right) + c$$

or

$$\frac{2}{\ln 10}\left(10^{\sqrt{x}}\right) + c$$

Graph of $y = b^x$, $b > 0$ and $b \neq 1$,

Range of $e^x > 0$
$(0, \infty)$

$f(x) = b^x = e^{x \ln b} > 0$ Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

For $\boxed{b > 1,}$ $\Rightarrow \ln b > 0$

$f(x) = b^x$

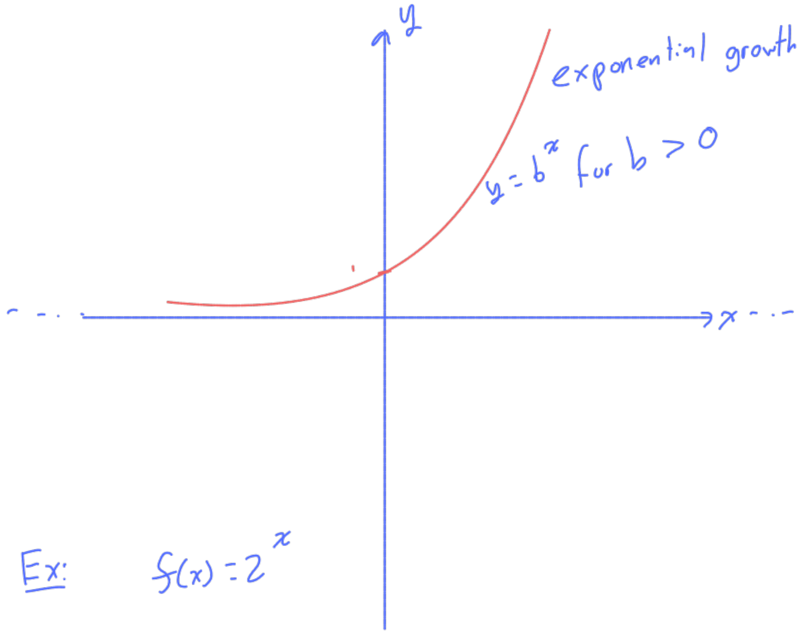
$f'(x) = \underbrace{b^x}_{>0} \underbrace{\ln b}_{>0} > 0 \Rightarrow f$ is always increasing

$f''(x) = b^x \ln b \cdot \ln b$

$= \underbrace{b^x}_{>0} \underbrace{(\ln b)^2}_{>0} > 0 \Rightarrow$ graph is concave up

$+ \; +$

$\cup$



Ex: $f(x) = 2^x$

$\lim_{x \to -\infty} (2^x) = 0$

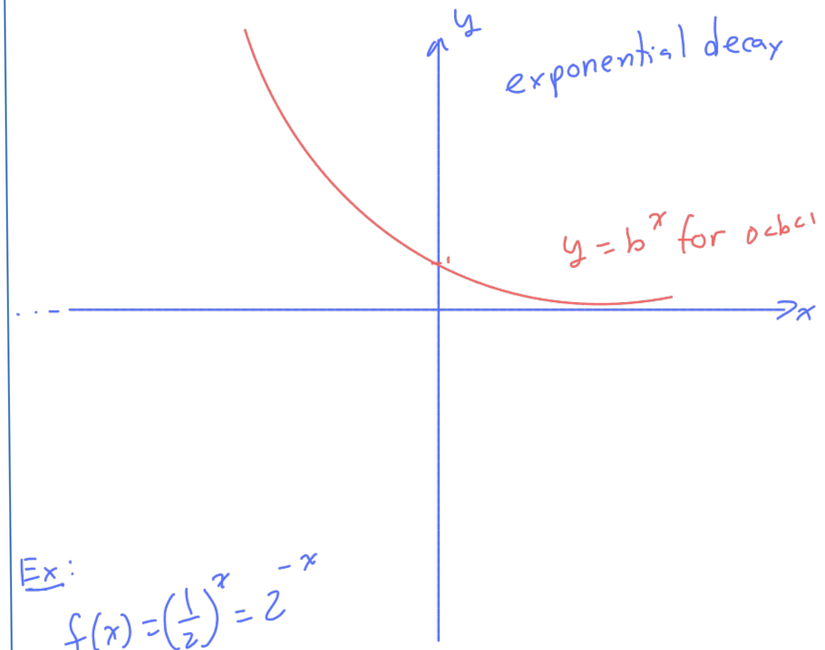
$\lim_{x \to -\infty} (2^x) = \infty$

For $0 < b < 1$,

$\boxed{\ln b < 0}$

$f'(x) = \underbrace{b^x}_{>0} \underbrace{\ln b}_{<0} < 0$ decreasing

$f''(x) = \underbrace{b^x}_{>0} \underbrace{(\ln b)^2}_{>0} > 0$ concave up.



Ex: $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$

$\lim_{x \to \infty} \left(\frac{1}{2}\right)^x = 0$

$\lim_{x \to -\infty} \left(\frac{1}{2}\right)^x = \infty$

General Logarithmic Functions

Since b^x is one-to-one, it has an inverse function. Its inverse function is called **the logarithmic function with base** b.

Definition of $\log_b x$

input ↓ output ↓

$$\log_b x = y \Leftrightarrow b^y = x \qquad \text{for } \boxed{b>0},\ b \neq 1,\ x>0$$

input output

$$b^{\boxed{y}} = \boxed{x}$$

*Note: $\ln x = \log_e x$ and $\log x = \log_{10} x$

Example: $\log_5(7) = y \Leftrightarrow 5^y = 7$

$$5^{\boxed{\log_5 7}} = 7$$

$$\log_{\text{base}}(\text{argument}) = \text{exponent} \Leftrightarrow \text{base}^{\text{exponent}} = \text{argument}$$

Inverse Function Cancellation equations:

x is in domain of b^x

x is in the domain of $\log_b x$

$$\log_b b^x = x \text{ for all } x \quad \text{and} \quad b^{\log_b x} = x \text{ for } x>0$$

① $\underbrace{\log_2 2^5}_{\log_2 32 = \boxed{5}} = 5$

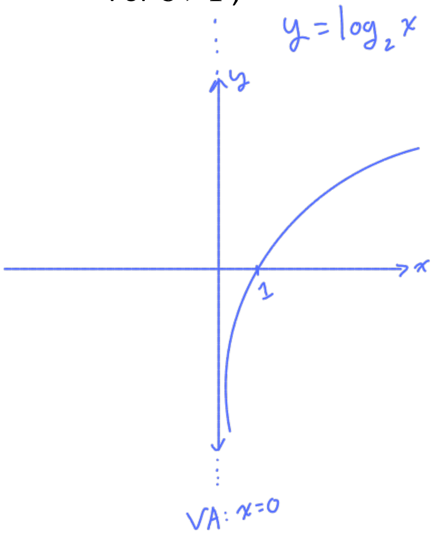
② $\log_7 49^3 = \log_7 (7^2)^3$
$= \log_7 7^6$
$= \boxed{6}$

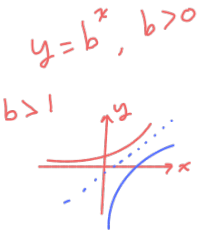
③ $28^{\log_{28}(17)} = \boxed{17}$

④ $8^{\log_2 5} = 2^{3\log_2 5} = 2^{\log_2 5^3} = 2^{\log_2 125} = \boxed{125}$

Graphs of logarithmic functions

For $b>1$,



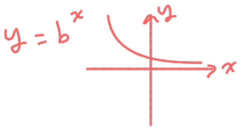


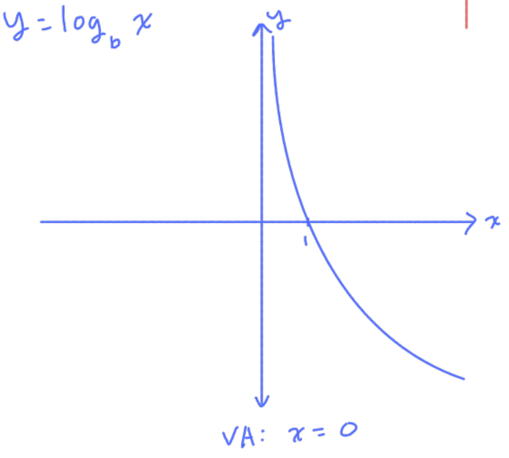
$y = \log_b x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

For $0<b<1$,





Laws of Logarithms

Quotient Law

Power Law

(1) $\log_b(xy) = \log_b x + \log_b y$
Product Law

(2) $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

(3) $\log_b\left(x^y\right) = y\log_b x$

Proof of (1): (**try-for-yourself**: prove the other two laws)

Let $u = \log_b x$ and $v = \log_b y$

Then $x = \boxed{b^u}$ and $y = \boxed{b^v}$

So:

$$\log_b(xy) = \log_b \boxed{\left(b^u b^v\right)} \quad \text{by substitution}$$

$$= \cancel{\log_b}\boxed{\cancel{\left(b^{u+v}\right)}} \quad \text{product law for } b^x$$

$$= u + v \quad \text{cancellation property}$$

$$= \log_b x + \log_b y \quad \text{by substitution}$$

$$\therefore \quad \log_b(xy) = \log_b x + \log_b y \qquad \text{Q.E.D.}$$

Change-of-base Formula:

$$\log_b(x) = \frac{\overset{\text{top}}{\log_a x}}{\underset{\text{bottom}}{\log_a(b)}}$$

In particular: $\boxed{\log_b x = \frac{\ln x}{\ln b}}$

example: $\log_{8} 16 = \frac{\log_2(16)}{\log_2(8)} = \frac{4}{3}$

$8^{\square} = 16$

$8^{4/3} = 16$

Proof: Proof

Let $y = \log_b x$

Then, by the definition of inverse functions,

$$x = \boxed{b^y}$$

Applying the natural log to both sides, we get

$$\ln x = \ln(b^y)$$

$$\frac{\ln x}{\ln b} = \frac{y\cancel{\ln b}}{\cancel{\ln b}} \Rightarrow y = \frac{\ln x}{\ln b}$$

$$\Rightarrow \log_b x = \frac{\ln x}{\ln b} \qquad \text{Q.E.D.}$$

Challenge

$$\frac{d}{dx}(\log_3 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 3}\right)$$

$$= \frac{1}{\ln 3}\cdot\frac{d}{dx}(\ln x)$$

$$= \frac{1}{\ln 3}\cdot\frac{1}{x}$$

$$= \boxed{\frac{1}{x\ln 3}}$$

Derivative of $\log_b x$

$$\frac{d}{dx}\log_b x = \frac{1}{\ln b}\cdot\frac{1}{x} = \frac{1}{x\ln b}$$

Chain Rule: $$\frac{d}{dx}\left(\log_b u\right) = \frac{1}{\ln b}\cdot\frac{u'}{u}$$

Proof: $$\frac{d}{dx}\log_b x = \frac{d}{dx}\boxed{\frac{\ln x}{\ln b}}$$ by the change of base formula

$$= \frac{1}{\ln b}\cdot\frac{d}{dx}\ln x$$

$$= \frac{1}{\ln b}\cdot\frac{1}{x}$$

$$= \frac{1}{x\ln b}$$

Example 3 Find the equation of the tangent line to the graph of $y = e^{\cos x}\log_3\left(3\tan x\right)$ at $x = \frac{\pi}{3}$.

Recall: The equation of a line with slope m and passing through the point $\left(x_1, y_1\right)$ is $\boxed{y - y_1 = m\left(x - x_1\right)}$

The equation of a **tangent line** to the graph of $y = f(x)$ at the point $x = a$ is given by

$(a, f(a))$

$$\boxed{y - f(a) = f'(a)\left(x - a\right)}$$

slope at that point $f'(a)$

Step 1: Find the derivative of the function.

$$y = \underbrace{e^{\cos x}}\cdot\log_3\left(3\tan x\right)$$

$$\frac{dy}{dx} = e^{\cos x}(-\sin x)\cdot\log_3(3\tan x) + e^{\cos x}\cdot\frac{1}{\ln 3}\cdot\frac{\cancel{3}\sec^2 x}{\cancel{3}\tan x}$$

$$= (-\sin x)e^{\cos x}\cdot\log_3(3\tan x) + \frac{e^{\cos x}\sec^2 x}{(\ln 3)\tan x}$$

$$\frac{d}{dx}\log_b x = \frac{d}{dx}\frac{\ln x}{\ln b}$$

$$= \frac{1}{\ln b}\cdot\frac{1}{x}$$

Step 2: Find the derivative of the function at the given point. (this is the slope of the tangent line at that point.), i.e. find $f'\left(a\right)$ with $a = \frac{\pi}{3}$ for this problem.

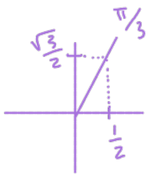
$$y'\left(\tfrac{\pi}{3}\right) = \left(-\sin\left(\tfrac{\pi}{3}\right)\right)e^{\cos\frac{\pi}{3}}\cdot\log_3\left(3\tan\tfrac{\pi}{3}\right) + \frac{e^{\cos\pi/3}\sec^2\frac{\pi}{3}}{(\ln 3)\tan\frac{\pi}{3}}$$

$$= \left(-\frac{\sqrt{3}}{2}\right)e^{1/2}\log_3\left(3\sqrt{3}\right) + \frac{e^{1/2}(2)^2}{(\ln 3)\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{2}\left(\sqrt{e}\right)\cancel{\log_3(3}^{3/2}) + \frac{4\sqrt{e}}{\sqrt{3}\ln 3}$$

$$= -\frac{\sqrt{3e}}{2}\cdot\frac{3}{2} + \frac{4\sqrt{e}}{\sqrt{3}\ln 3} = \boxed{\frac{-3\sqrt{3e}}{4} + \frac{4\sqrt{e}}{\sqrt{3}\ln 3}}$$

$\frac{1}{2}, \frac{\sqrt{3}}{2}$



$3\sqrt{3}$

$= 3^1\cdot 3^{1/2}$

$= 3^{1+\frac{1}{2}} = 3^{3/2}$

Example 3 cont'd:

Find the <u>equation of the **tangent line**</u> to the graph of $y = e^{\cos x}\log_3(3\tan x)$ at $x = \frac{\pi}{3}$

The equation of a **tangent line** to the graph of $y = f(x)$ at the point $x = a$ is given by

$$\boxed{y - f(a) = f'(a)(x - a)}$$

$$y - y_1 = m(x - x_1)$$

<u>Step 3</u>: Find the equation of the tangent line using the point and the slope of the tangent line at that point. (if it's not given, you'll need to first find the y-value at the given point, i.e. find $f(a)$)

$$f\left(\frac{\pi}{3}\right) = e^{\cos\frac{\pi}{3}} \cdot \log_3\left(3\tan\pi/3\right)$$

$$= e^{1/2}\log_3(3\sqrt{3})$$

$$= \sqrt{e} \cdot \frac{3}{2} = \frac{3\sqrt{e}}{2}$$

<u>EQUATION:</u>

$$\boxed{y - \frac{3\sqrt{e}}{2} = \left(\frac{-3\sqrt{3e}}{4} + \frac{4\sqrt{e}}{\sqrt{3}\ln 3}\right)\left(x - \frac{\pi}{3}\right)}$$