Error Function

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What is an Error Function?

The Error Function is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t \tag{1}$$

Graph of $\operatorname{erf}(x)$:

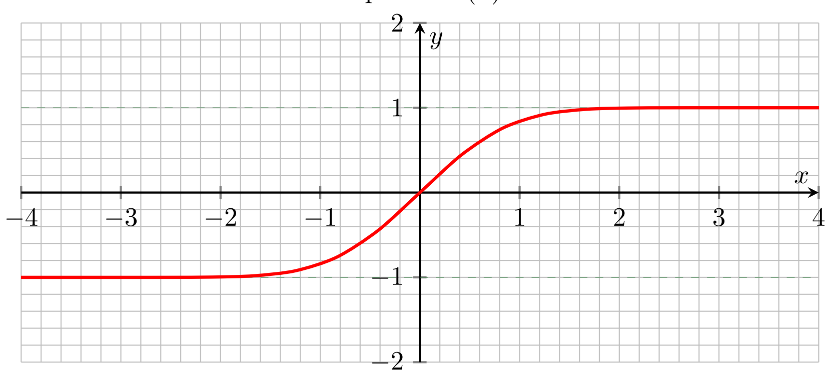

Figure 1: Graph of the Error Function $\operatorname{erf}(x)$.

Now you can see, this is an odd function! So $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ As shown in Figure 1, the graph also has horizontal asymptotes of $y = 1$ and $y = -1$.

There is also a complementary error function:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,\mathrm{d}t \tag{2}$$

There's an interesting property that arises when we combine these two error functions:

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1 \tag{3}$$

Now, why would we care about this? We can't solve the integral but it appears frequently in statistics, heat transfer, and physics, so we gave it a function name and manually computed its values for later reference.

Now that you know what it is and how the graph looks, let's try some problems!

First split it apart:

$$\int_0^t x^2 e^{-x^2}\,\mathrm{d}x = \int_0^t x \cdot x e^{-x^2}\,\mathrm{d}x \tag{1}$$

Then, use integration by parts. Let

$$u = x, \qquad \mathrm{d}v = x e^{-x^2}\,\mathrm{d}x.$$

Then (using u-sub for $\mathrm{d}v \to v$)

$$\mathrm{d}u = \mathrm{d}x, \qquad v = \int x e^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}.$$

Thus,

$$\int x^2 e^{-x^2}\,\mathrm{d}x = uv - \int v\,\mathrm{d}u = -\frac{x}{2}e^{-x^2} + \frac{1}{2}\int e^{-x^2}\,\mathrm{d}x + C.$$

Since

$$\int e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x),$$

the anti-derivative becomes

$$\int x^2 e^{-x^2}\,\mathrm{d}x = -\frac{x}{2}e^{-x^2} + \frac{\sqrt{\pi}}{4}\,\mathrm{erf}(x) + C.$$

Evaluating from 0 to t:

$$\int_0^t x^2 e^{-x^2}\,\mathrm{d}x = \left[-\frac{x}{2}e^{-x^2} + \frac{\sqrt{\pi}}{4}\,\mathrm{erf}(x) + C\right]_0^t.$$

Since $e^0 = 1$, $\mathrm{erf}(0) = 0$ and the C's cancel, the result is

$$\boxed{\int_0^t x^2 e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{4}\,\mathrm{erf}(t) - \frac{t}{2}e^{-t^2}}.$$

2 $\int \operatorname{erf}(x)\,\mathrm{d}x$

Start with integration by parts.

$$u = \operatorname{erf}(x), \qquad \mathrm{d}v = \mathrm{d}x.$$

Use the derivative of $\operatorname{erf}(x)$ from the previous problem to solve.

$$\mathrm{d}u = \frac{2}{\sqrt{\pi}} e^{-x^2}\,\mathrm{d}x, \qquad v = x$$

Thus,

$$\int \operatorname{erf}(x)\,\mathrm{d}x = uv - \int v\,\mathrm{d}u = x\operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \int x e^{-x^2}\,\mathrm{d}x$$

Using the integral of $\int x e^{-x^2}\,\mathrm{d}x$ from the previous problem, the result is

$$\boxed{\int \operatorname{erf}(x)\,\mathrm{d}x = x\operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + C}.$$

3 $\displaystyle\int_1^t \frac{1}{e^{x^2}\,\mathrm{erf}(x)}\,\mathrm{d}x$

First, apply u-sub

$$\text{Let: } u = \mathrm{erf}(x), \qquad \mathrm{d}u = \frac{2}{\sqrt{\pi}}e^{-x^2}\,\mathrm{d}x$$

Then, isolate $\mathrm{d}x$

$$\mathrm{d}u = \frac{2}{\sqrt{\pi}}e^{-x^2}\,\mathrm{d}x \rightarrow \frac{\sqrt{\pi}}{2}e^{x^2}\,\mathrm{d}u = \mathrm{d}x$$

Thus,

$$\int_1^t \frac{1}{e^{x^2}\,\mathrm{erf}(x)}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\int \frac{1}{u}\,\mathrm{d}u, \qquad u = erf(x)$$

Evalutate and plug back in your u

$$\int_1^t \frac{1}{e^{x^2}\,\mathrm{erf}(x)} = \frac{\sqrt{\pi}}{2}\left[\ln(\mathrm{erf}(x))\right]_1^t$$

Plug in your bounds and simplify. The result is

$$\boxed{\int_1^t \frac{1}{e^{x^2}\,\mathrm{erf}(x)} = \frac{\sqrt{\pi}}{2}\ln\left[\frac{\mathrm{erf}(t)}{\mathrm{erf}(1)}\right]}.$$

4 $\displaystyle\int \frac{2e^{-x^2}}{\operatorname{erfc}(x)-\operatorname{erf}(x)}\,\mathrm{d}x$

First, notice the denominator can be modified with the error function identity.

$$\operatorname{erfc}(x)+\operatorname{erf}(x)=1 \to \operatorname{erfc}(x)=1-\operatorname{erf}(x)$$

Thus,

$$\int \frac{2e^{-x^2}}{\operatorname{erfc}(x)-\operatorname{erf}(x)}\,\mathrm{d}x=\int \frac{2e^{-x^2}}{1-2\operatorname{erf}(x)}\,\mathrm{d}x$$

Now, you can apply u-sub

$$\text{Let: } u=1-2\operatorname{erf}(x), \qquad \mathrm{d}u=-2\frac{2}{\sqrt{\pi}}e^{-x^2}\,\mathrm{d}x$$

Then, isolate $\mathrm{d}x$.

$$\mathrm{d}u=-2\frac{2}{\sqrt{\pi}}e^{-x^2}\,\mathrm{d}x \to -\frac{\sqrt{\pi}}{4}e^{x^2}\,\mathrm{d}u$$

Thus,

$$\int \frac{2e^{-x^2}}{\operatorname{erfc}(x)-\operatorname{erf}(x)}\,\mathrm{d}x=-\frac{\sqrt{\pi}}{2}\int \frac{1}{u}\,\mathrm{d}u, \qquad u=1-2\operatorname{erf}(x)$$

Integrate

$$\int \frac{2e^{-x^2}}{\operatorname{erfc}(x)-\operatorname{erf}(x)}\,\mathrm{d}x=-\frac{\sqrt{\pi}}{2}\ln|u|+C, \qquad u=1-2\operatorname{erf}(x)$$

Finally, plug back in your u.

$$\boxed{\int \frac{2e^{-x^2}}{\operatorname{erfc}(x)-\operatorname{erf}(x)}\,\mathrm{d}x=-\frac{\sqrt{\pi}}{2}\ln|1-2\operatorname{erf}(x)|+C}$$

5 $\displaystyle\int \frac{2\ln\left(\mathrm{erf}(x)e^{-x^2}\right)\left(\frac{1}{\sqrt{\pi}}e^{-x^2} - x\cdot\mathrm{erf}(x)\right)}{\mathrm{erf}(x)}\,\mathrm{d}x$

First, recognize that the numerator's second term looks similar to product rule. Try u-sub with the natural log's argument:

$$\text{Let: } u = \mathrm{erf}(x)e^{-x^2}, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2e^{-x^2}\left[\frac{1}{\sqrt{\pi}}e^{-x^2} - x\cdot\mathrm{erf}(x)\right]$$

Now, we can let $\frac{\mathrm{d}u}{\mathrm{d}x} = u'$ (for easier viewing) and isolate that second term in the numerator

$$\frac{u'}{2e^{-x^2}} = \frac{1}{\sqrt{\pi}}e^{-x^2} - x\cdot\mathrm{erf}(x)$$

If we take then take our defined u and isolate $e^{-}x^2$,

$$e^{-x^2} = \frac{u}{\mathrm{erf}(x)} \to \frac{u'}{2}\cdot\frac{\mathrm{erf}(x)}{u} = \frac{1}{\sqrt{\pi}}e^{-x^2} - x\cdot\mathrm{erf}(x)$$

Substituting back in our $\frac{\mathrm{d}u}{\mathrm{d}x}$ and moving $\mathrm{d}x$ to the right side,

$$\frac{\mathrm{d}u}{2u} = \frac{\frac{1}{\sqrt{\pi}}e^{-x^2} - x\cdot\mathrm{erf}(x)}{\mathrm{erf}(x)}\,\mathrm{d}x$$

Now, substituing back into the original equation, such that

$$\int \frac{2\ln\left(\mathrm{erf}(x)e^{-x^2}\right)\left(\frac{1}{\sqrt{\pi}}e^{-x^2} - x\cdot\mathrm{erf}(x)\right)}{\mathrm{erf}(x)}\,\mathrm{d}x = \int \frac{\ln(u)}{u}\,\mathrm{d}u, \qquad u = \mathrm{erf}(x)e^{-x^2}$$

Now, use a simple v-substitution

$$\text{Let: } v = \ln(u), \qquad \mathrm{d}v = \frac{1}{u}\,\mathrm{d}u$$

Thus,

$$\int v\,\mathrm{d}v = \frac{1}{2}v^2 + C$$

Finally, plug your v in, then your u

$$\frac{1}{2}v^2 + C \to \frac{1}{2}\ln(u)^2 + C \to \frac{1}{2}\ln\left|\mathrm{erf}(x)e^{-x^2}\right|^2 + C$$

This leaves you with the result

$$\boxed{\int \frac{2\ln\left(\mathrm{erf}(x)e^{-x^2}\right)\left(\frac{1}{\sqrt{\pi}}e^{-x^2} - x\cdot\mathrm{erf}(x)\right)}{\mathrm{erf}(x)}\,\mathrm{d}x = \frac{1}{2}\ln\left|\mathrm{erf}(x)e^{-x^2}\right|^2 + C}$$