Feynman's Integration Technique

By SMC Integration Club

Feynman's Integration Technique (also known as Differentiation under the Integral Sign and the Leibniz Integral Rule) is a method of simplifying difficult integrals by using partial differentiation of an extra variable (or parameter).

First, the basic idea. Suppose you have an integral:

$$I = \int_a^b f(x)\,\mathrm{d}x$$

If you're unsure of how to solve it directly, we can use Feynman's technique. This involves introducing an extra variable, usually t, called a **parameter**. Your integral will become:

$$I(t) = \int_a^b f(x,t)\,\mathrm{d}x$$

Now, instead of integrating with respect to x, we'll differentiate with respect to t, which can simplify the expression within the integral.

Now, we'll go over an example. Consider the following:

$$I = \int_0^1 \frac{x-1}{\ln(x)}\,\mathrm{d}x$$

This isn't the most pleasant integral. Standard substitution and integration by parts are dead ends. Let's try introducing a parameter. In this case, we'll replace the exponent of x with t:

$$I(t) = \int_0^1 \frac{x^t-1}{\ln(x)}\,\mathrm{d}x$$

So, we need to define some parameters:

- Target Parameter: This is the value of t that will reproduce the original integral. In this example, $t = 1$ will reproduce the original integral.
- Initial Parameter: This is a value of t that makes the parameterized integral easy to evaluate. In this case, we'll use $t = 0$ because that would give us $I(0) = 0$.
- Parameter Domain: This is the domain of t where the parameterized integral is valid. This is important to consider because the bounds seen on the original intergral only apply to x (for this problem), so when you integrate with respect to t, you need to ensure the solution exists.

We want to find an initial parameter to recover the value of the integral at the target parameter. So, because we're going to use $t = 1$, let's differentiate with respect to t.

$$I'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{x^t - 1}{\ln(x)} \, \mathrm{d}x$$

Now, move the derivative inside the integral:

$$I'(t) = \int_0^1 \frac{\partial}{\partial t} \left(\frac{x^t - 1}{\ln x} \right) \, \mathrm{d}x$$

Recall your power rule of differentiation.

$$\frac{\partial}{\partial t} x^t = x^t \ln(x)$$

Therefore,

$$I'(t) = \int_0^1 \frac{x^t \ln(x)}{\ln(x)} \, \mathrm{d}x$$

Now, when we simplify, the two $\ln(x)$ cancel, leaving you with a simpler integral.

$$I'(t) = \int_0^1 x^t \, \mathrm{d}x$$

So, evaluate the definite integral,

$$I'(t) = \left[\frac{x^{t+1}}{t+1} \right]_0^1$$

Normally, you'd evaluate this directly, but we need to define a domain for our t here. This is because, 0^{t+1} doesn't always equal zero. If we were to test $t = 2$:

$$\int_0^1 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

At the lower bound, $0^3 = 0$; great. But, at $t = -2$:

$$\int_0^1 x^{-2} \, \mathrm{d}x = \int_0^1 \frac{1}{x^2} \, \mathrm{d}x$$

The antiderivative is $-\frac{1}{x}$, which will grow without bound as $x \to 0^+$ because we're evaluating it as a limit instead of just substituting it right in. We **need** something that converges. So, for the lower bound term to approach 0, we need the exponent to be positive.

$$t + 1 > 0 \Longrightarrow t > -1$$

Do remember that you'll also need to ensure that the denominator doesn't become 0 as well, meaning we can see that t cannot be -1 itself, before ever evaluating a full domain. Now,

$$\text{Given } t > -1, \qquad I'(t) = \frac{1}{t+1}$$

But, this isn't our answer, because we want $I(t)$, not $I'(t)$. So, let's integrate with respect to t as an indefinite integral in an attempt to recover the original function:

$$I(t) = \int \frac{1}{t+1} \, \mathrm{d}t = \ln(t+1) + C$$

Since we integrated indefinitely, we now have a constant of integration. To find it out, we'll look at the parameterized integral when $t = 0$, our initial parameter. Remember, we chose this value because it made the numerator 0.

$$I(0) = \int_0^1 \frac{x^0 - 1}{\ln(x)} \, \mathrm{d}x$$

Since $x^0 = 1$,

$$I(0) = 0 = \ln(0+1) + C = C$$

Now that we know C, we can substitute in our target parameter, $t = 1$ to recover the original integral:

$$I(1) = \ln(1+1) = \ln(2)$$

Let's try some problems now:

1) $\displaystyle\int_0^1 \frac{x^3-1}{\ln(x)}\,\mathrm{d}x$

2) $\displaystyle\int_0^1 \frac{x^2 - x^5}{\ln(x)}\,\mathrm{d}x$

3) $\displaystyle\int_0^{\infty} \frac{e^{-x} - e^{-3x}}{x}\,\mathrm{d}x$

4) $\displaystyle\int_0^\infty e^{-2x}\frac{\sin(x)}{x}\,\mathrm{d}x$

5) $\displaystyle\int_0^\infty e^{-x}\,\frac{\sin(3x)-\sin(x)}{x}\,\mathrm{d}x$