Floor Function

By SMC Integration Club

Before we start, let's have a quick review of series.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

The Test for Divergence (or the nth-term test):

If $\lim\limits_{n\to\infty} a_n \neq 0$ (or the limit does not exist or is $\pm\infty$), then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

The Geometric Series:

Given the **geometric series** with ratio r and first term $a \neq 0$:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \dots$$

(i) If $|r| < 1$, then the series is convergent and its sum is $\sum\limits_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

(ii) If $|r| \geq 1$, then the series diverges.

Also, don't forget about your **comparison tests**!

Now...

What is the Floor Function?

Simply, a floor function returns an integer that is less than or equal to the inputted number. It is strictly defined as:

$$\lfloor x \rfloor = \max \left\{ m \in \mathbb{Z} | m \leq x \right\} \tag{1}$$

Graph of $\lfloor x \rfloor$:

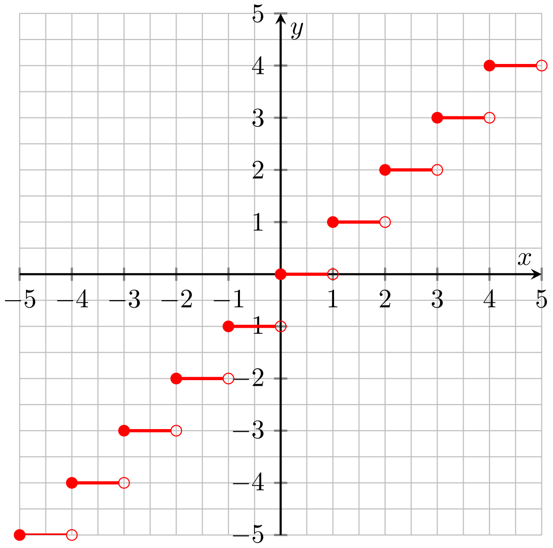

Figure 1: Graph of the Floor Function $\lfloor x \rfloor$.

Some examples of the floor function:

$\lfloor 5.9999 \rfloor = 5$
$\lfloor -0.5 \rfloor = -1$
$\lfloor 1 \rfloor = 1$
$\lfloor 0.7 \rfloor = 0$
$\lfloor -1.34 \rfloor = -2$

The floor function's derivative is simple, especially when you can just analyze the graph.

$$\frac{\mathrm{d}}{\mathrm{d}x}\lfloor x\rfloor = \begin{cases} 0, & x \notin \mathbb{Z} \\ \text{does not exist}, & x \in \mathbb{Z} \end{cases} \tag{2}$$

If you want a peek at the distributional derivative, it's the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\lfloor x\rfloor = \sum_{k\in\mathbb{Z}} \delta(x-k) \tag{3}$$

All this says is that where $x = k$, there is a spike and the derivative does not exist. Anywhere else is 0. You won't need to know this though, for now.

Now, the important part: Integrals!
The integral of the floor function can be extremely simple or a bit long depending on what restrictions you apply. For example,

$$\int_0^n \lfloor x\rfloor \,\mathrm{d}x = \sum_{k=0}^{n-1} \int_k^{k+1} k \,\mathrm{d}x = \frac{n(n-1)}{2}, \qquad \text{for } n \in \mathbb{Z} \text{ and } n \geq 1 \tag{4}$$

But look, the one above only works for n being an integer and only if it is greater than 0. So, what if we wanted to include 0? Or negative numbers?

$$\int_a^b \lfloor x\rfloor \mathrm{d}x = \frac{\lfloor b\rfloor(\lfloor b\rfloor - 1)}{2} - \frac{\lfloor a\rfloor(\lfloor a\rfloor - 1)}{2} + \lfloor b\rfloor(b - \lfloor b\rfloor) - \lfloor a\rfloor(a - \lfloor a\rfloor) \tag{5}$$

The one above is a floor specific formula. But, a general interval decomposition identity will be useful as well:

$$\int_a^b f(x)\,\mathrm{d}x = \int_{\lfloor a\rfloor}^{\lfloor b\rfloor} f(x)\,\mathrm{d}x + \int_{\lfloor b\rfloor}^{b} f(x)\,\mathrm{d}x - \int_{\lfloor a\rfloor}^{a} f(x)\,\mathrm{d}x \tag{6}$$

Now, Equation (5) looks pretty ugly, I know. The reason is because this version allows non-integers and zero in a non-integral form. Let's break it down; we'll use t as a dummy variable here.

$$F(x) = \int_0^x \lfloor t\rfloor \,\mathrm{d}t, \quad \longrightarrow \quad \int_a^b f(x)\,\mathrm{d}x = \int_0^b f(x)\,\mathrm{d}x - \int_0^a f(x)\,\mathrm{d}x$$

Now,

$$\text{When } f(x) = \lfloor x\rfloor, \qquad \int_a^b \lfloor x\rfloor \,\mathrm{d}x = F(b) - F(a)$$

So, with this we've only defined $F(x)$ implicitly as an integral, so how can we write $F(x)$ without an integral sign? Let's think about this geometrically.

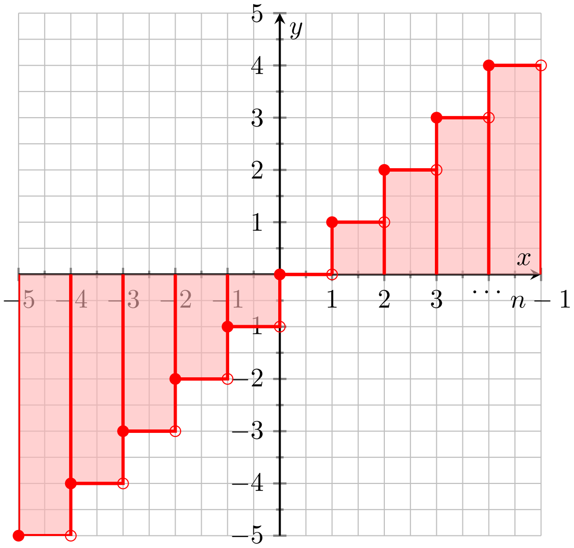

Figure 2: Shaded graph of the Floor Function $\lfloor x \rfloor$.

The graph shows that, the integral of the floor function is really just adding up multiple rectangles. If a bound is a non-integer, it's really just a smaller rectangle. So,

$$\text{Full rectangles from 0 to } n\text{: } \frac{n(n-1)}{2}, \quad n = \lfloor x \rfloor$$

$$\text{Partial rectangles from } n \text{ to } x\text{: } n(x-n), \quad n = \lfloor x \rfloor$$

Thus,

$$F(x) = \int_0^x \lfloor t \rfloor \, \mathrm{d}t = \frac{n(n-1)}{2} + n(x-n), \quad n = \lfloor x \rfloor$$

But, we want from some initial value that may not always be 0. So, by using the $F(b) - F(a)$ mentioned before, we can assign each version it's expanded form and just subtract the two.

$$\int_a^b \lfloor x \rfloor \, \mathrm{d}x = \left[\frac{\lfloor b \rfloor (\lfloor b \rfloor - 1)}{2} + \lfloor b \rfloor (b - \lfloor b \rfloor) \right] - \left[\frac{\lfloor a \rfloor (\lfloor a \rfloor - 1)}{2} + \lfloor a \rfloor (a - \lfloor a \rfloor) \right]$$

That's where you get:

$$\boxed{\int_a^b \lfloor x \rfloor \mathrm{d}x = \frac{\lfloor b \rfloor (\lfloor b \rfloor - 1)}{2} - \frac{\lfloor a \rfloor (\lfloor a \rfloor - 1)}{2} + \lfloor b \rfloor (b - \lfloor b \rfloor) - \lfloor a \rfloor (a - \lfloor a \rfloor)} \quad (7)$$

Now that we've got a handful of methods to integrate a floor function, let's try some problems now:

1) $\displaystyle\int\limits_0^\infty \lfloor x \rfloor \, \mathrm{d}x$

The previous example isn't too interesting, we know it'll go to infinity. A better example that pulls in summation formulas requires a more restrictive domain.

2) $\displaystyle\int\limits_0^{10} \lfloor x \rfloor \, \mathrm{d}x$

A bit more complicated, but it's possible given what you've been taught.

3) $\displaystyle\int\limits_0^\infty \mathrm{e}^{-\lfloor x \rfloor}\,\mathrm{d}x$

Will the following integrals **converge** or **diverge**?

4) $$\int_{1}^{\infty} \frac{\lfloor x \rfloor}{x} \, \mathrm{d}x$$

5) $\displaystyle\int_{2}^{\infty} \frac{\lfloor x \rfloor x^2}{\left(x^3+1\right)^2}\,\mathrm{d}x$

For problems **6** and **7**, simplify completely within reason. You may leave any summations in summation form.

6) $$\int_{0.5}^{100.5} x^2 e^{-\lfloor x \rfloor}\, \mathrm{d}x$$

7) $$\int_{0.5}^{7.5} x e^{-\lfloor x \rfloor}\, \mathrm{d}x$$

Last one! It's challenging, but you can find what this "converged" to! Remember: Some sums tends to collapse into themselves after expansion.

8) $$\int_{1}^{\infty} \frac{1}{\lfloor x \rfloor x^2} \, \mathrm{d}x$$