Gamma Function

by Thomas Yamada

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What is the Gamma Function?

The Gamma Function is defined as:

$$\Gamma(z) = \int_{u=0}^{\infty} u^{z-1} e^{-u} \, \mathrm{d}t \tag{1}$$

Graph of $\Gamma(z)$:

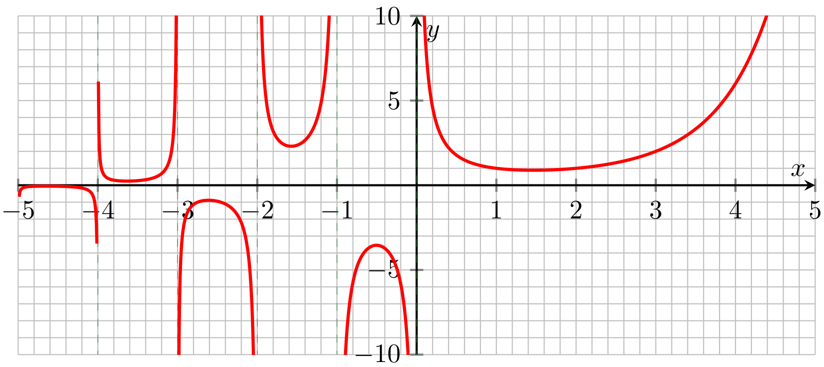

Figure 1: Graph of the Gamma Function $\Gamma(z)$.

You can see, by the graph, that the function is neither odd or even, because its asymmetric everywhere. There is no linearity either.

There is two relations though. It's core identity is the recurrence relation, the Gamma's equivalent of a "linearity rule":

$$\Gamma(z+1) = z\Gamma(z) \tag{2}$$

Then, you have Euler's Reflection Formula, a sort of complementary behavior:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{3}$$

So, what is this Gamma Function really? Remember the factorial? Well,

$$\Gamma(z+1) = z! \tag{4}$$

It's partially that simple, but also not exactly. If you remember using the factorial, you've only ever used integers, because that's easy to compute with the basic function. i.e. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

But, how would you do that with $\frac{1}{2}!$? What about $-\frac{1}{2}!$? That's where the Gamma Function steps in. It allows you to computer the "factorial" of non integers. It does have some restrictions though:

$$\mathrm{Dom}(\Gamma) = \mathbb{R} \setminus \left\{ -n : n \in \mathbb{Z}_{\geq 0} \right\} \tag{5}$$

Let's try $\Gamma(1)$:

$$\begin{aligned} \Gamma(1) &= \int\limits_{u=0}^{\infty} u^0 e^{-u} \, \mathrm{d}u = \int\limits_{u=0}^{\infty} e^{-u} \, \mathrm{d}u \\ \Gamma(1) &= \left[-e^{-u} \right]_{u=0}^{\infty} = 0 - (-1) \\ \Gamma(1) &= 1 = 0! \end{aligned} \tag{6}$$

For $\Gamma(2)$, you'll need to use a simple IBP:

$$\begin{aligned} \Gamma(2) &= \int\limits_{u=0}^{\infty} u^1 e^{-u} \, \mathrm{d}u = \int\limits_{u=0}^{\infty} t e^{-u} \, \mathrm{d}u \\ \Gamma(2) &= -ue^{-u} + \int\limits_{u=0}^{\infty} e^{-u} \, \mathrm{d}u = 0 + 1 \\ \Gamma(2) &= 1 = 1! \end{aligned} \tag{7}$$

You can probably see where this is going. $\Gamma(3) = 2 = 2!$

So, what about the recurrence relation? How can we use that? Well, keep in mind that computing $\Gamma(0)$ does not exist, as you can see from the graph above. So, if we were to apply $\Gamma(z+1) = z\Gamma(z)$ to get, for example, a negative integer, you'd need to re-use the relation again and again until you reached a computable number, but you'd always hit 0 before anything actually computable, which is why every negative integer less than 0 does not exist.

Then, what about negative fractionals? Let's use some negative fractional that is perfectly divisible by $\frac{1}{2}$, such as $-\frac{3}{2}$. You'll need to use the recurrence relation in order to get to that. The first directly computable value for that is $\Gamma(\frac{1}{2})$, so we need to solve that first. Luckily, it gives a nice number to work with.

Let's look into how we'd find $\Gamma\left(\frac{1}{2}\right)$:

$$\Gamma\left(\frac{1}{2}\right) = \int\limits_{u=0}^{\infty} e^{-u} u^{-\frac{1}{2}} \, \mathrm{d}u$$

Let's rewrite it so it's easier to read.

$$\Gamma\left(\frac{1}{2}\right) = \int\limits_{u=0}^{\infty} e^{-u} \frac{1}{\sqrt{u}} \, \mathrm{d}u$$

Now, let's apply u-sub,

$$\text{Let: } w = \sqrt{u}, \qquad \mathrm{d}w = \frac{1}{2\sqrt{u}} \, \mathrm{d}w, \qquad u = w^2$$

Next, substitute back in and factor out the 2.

$$\Gamma\left(\frac{1}{2}\right) = 2 \int\limits_{w=0}^{\infty} e^{-w^2} \, \mathrm{d}w$$

Looking at the graph for this new function, we can see that the integral can consume the 2 and expand the integral bounds from $0 \to \infty$ to $-\infty \to \infty$.

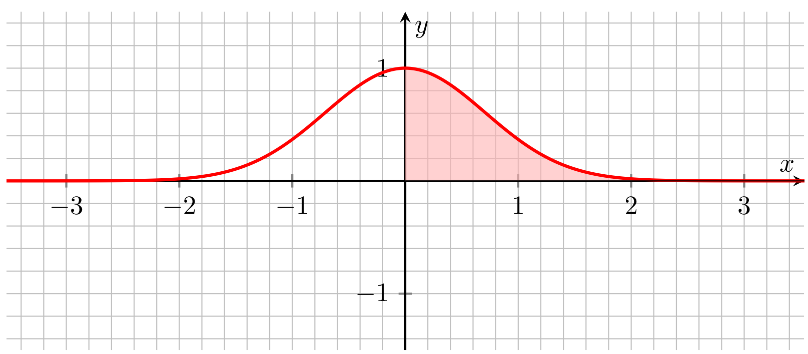

Figure 2: Graph of the Bell Curve $f(x) = e^{-x^2}$.

So, let's change the bounds. Let's also set the integral to some arbitrary variable, J.

$$J = \int\limits_{w=-\infty}^{\infty} e^{-w^2} \, \mathrm{d}w$$

Since e^{-w^2} has no elementary anti derivative, we can use a Mutlivariable Calculus trick to solve the integral. First, we'll square both sides, splitting the right hand side into two integrals with arbitrary variables, x and y.

$$J^2 = \left(\int\limits_{x=-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x \right) \left(\int\limits_{y=-\infty}^{\infty} e^{-y^2} \, \mathrm{d}y \right)$$

Now, we need to combine them. We'll do that using Fubini's Theorem, turning the squared/multiplied integrals into a double integral.

$$J^2 = \int\limits_{x=-\infty}^{\infty} \int\limits_{y=-\infty}^{\infty} e^{-x^2} e^{-y^2} \, \mathrm{d}y \, \mathrm{d}x$$

Looking at this, the x and y covers the entire Cartesian Plane. So, we can compact the double integral to cover all real numbers on the 2D plane, $\mathbb{R}^2$.

$$J^2 = \iint\limits_{\mathbb{R}^2} e^{-x^2} e^{-y^2} \, \mathrm{d}x \, \mathrm{d}y = \iint\limits_{\mathbb{R}^2} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y$$

The exponent on the e looks similar to the circle equation, so we're going to convert the integral into polar coordinates.

$$r^2 = x^2 + y^2 \qquad \mathrm{d}x \, \mathrm{d}y = r \, \mathrm{d}r \, \mathrm{d}\theta$$

The angle only needs to go from $0 \to 2\pi$ to cover the entire graph, as well as $0 \to \infty$ for the radius.

$$J^2 = \int\limits_{\theta=0}^{2\pi} \int\limits_{r=0}^{\infty} e^{-r^2} r \, \mathrm{d}r \, \mathrm{d}\theta$$

Now, we can split this apart again by doing a reverse Fubini's Theorem and solving the left, simple, integral.

$$J^2 = \left(\int\limits_{\theta=0}^{2\pi} \mathrm{d}\theta \right) \left(\int\limits_{r=0}^{\infty} e^{-r^2} r \, \mathrm{d}r \right) = 2\pi \int\limits_{r=0}^{\infty} e^{-r^2} r \, \mathrm{d}r$$

From here, do u-sub letting $v = r^2$. We're using v because u is already used before in the problem.

$$\text{Let: } v = r^2, \qquad \mathrm{d}v = 2r \, \mathrm{d}r$$

Substituting it in, the 2's and r's cancel.

$$J^2 = \pi \int\limits_{r=0}^{\infty} e^{-v} \, \mathrm{d}v$$

Next, solve the integral and evaluate the bounds of integration.

$$J^2 = \pi \left[\frac{e^{-v}}{-1} \right]_{r=0}^{\infty} = \pi \left[0 - (-1) \right]$$

Finally, simplify in the brackets and take the square root of both sides, isolating J.

$$J^2 = \pi \quad \to \quad J = \sqrt{\pi}$$

Substituting back in the original function for J, we get:

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

Now, we've gone over the graph of the function as well as a relatively complex example. Let's do some problems!

First, let's convert it into terms of the Gamma Function.

1 $\frac{3}{2}! = \Gamma(5/2)$

Now, let's split it apart and pull the z out,

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

So, you've lowered the value in the Gamma Function, but its not yet low enough to be something we've seen before. Let's do it again.

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

Oh, we've seen that before! $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Thus,

$$\boxed{\frac{3}{2}! = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}}$$

Before anything, let's expand $(x+1)^4$,

$$\mathbf{2} \quad \int_0^\infty e^{-x}(x+1)^4 \,\mathrm{d}x = \int_0^\infty e^{-x} \cdot (x^4 + 4x^3 + 6x^2 + 4x + 1) \,\mathrm{d}x$$

Now, distribute and split apart the integral(s). I'm also going to assign the original integral the variable R:

$$R = \int_0^\infty x^4 e^{-x} \,\mathrm{d}x + 4\int_0^\infty x^3 e^{-x} \,\mathrm{d}x + 6\int_0^\infty x^2 e^{-x} \,\mathrm{d}x + 4\int_0^\infty x e^{-x} \,\mathrm{d}x + \int_0^\infty \,\mathrm{d}x$$

Remember the simple Gamma Function of a Factorial, that is:

$$n! = \int_0^\infty e^{-u} u^n \,\mathrm{d}u$$

So, if we switch each integral into factorial form,

$$R = 4! + 4 \cdot 3! + 6 \cdot 2! + 4 \cdot 1! + 0!$$

Computing this gives you your answer!

$$\boxed{R = 65}$$

3 $\displaystyle\int_1^\infty \frac{(\ln(x))^3}{x^2}\,\mathrm{d}x$

Let's u-sub first.

$$\text{Let: } u = \ln(x) \quad\implies\quad x = e^u, \qquad \mathrm{d}x = e^u\,\mathrm{d}u$$

Before substituting it back in, we need to change the bounds of integration,

$$\text{When: } x = 1 \to u = \ln(1) = 0$$

$$\text{When: } x = \infty \to u = \ln(\infty) = \infty$$

We can now substitute everything in,

$$\int_1^\infty \frac{(\ln(x))^3}{x^2}\,\mathrm{d}x = \int_0^\infty \frac{u^3}{e^{2u}} \cdot e^u\,\mathrm{d}u$$

We can now cancel things and move values up to the numerator, since we're looking for the Gamma Function form.

$$\int_1^\infty \frac{(\ln(x))^3}{x^2}\,\mathrm{d}x = \int_0^\infty e^3 e^{-u}\,\mathrm{d}u$$

That's a recognizable form!

$$\boxed{\int_1^\infty \frac{(\ln(x))^3}{x^2}\,\mathrm{d}x = 3! = 6}$$

4 $\int_0^\infty x^6 e^{-4x^2}\,\mathrm{d}x$

First step, we'll perform a u-sub,

$$\text{Let: } u = 4x^2, \qquad \frac{\mathrm{d}u}{8x} = \mathrm{d}x, \qquad x^2 = \frac{u}{4}, \qquad x = \frac{\sqrt{u}}{2}$$

Don't forget to also check the bounds,

$$\text{When: } x = 0 \Longrightarrow u = 0, \qquad x = \infty \Longrightarrow u = \infty$$

We also need to split apart the x^6 to a defined value we already have.

$$x^6 = \left(x^2\right)^3, \qquad x^6 = \left(\frac{u}{4}\right)^3 = \frac{u^3}{64}$$

There we go! Let's substitute in!

$$\int_0^\infty x^6 e^{-4x^2}\,\mathrm{d}x = \int_0^\infty \frac{u^3}{64} e^{-u} \frac{1}{4\sqrt{u}}\,\mathrm{d}u = \frac{1}{256}\int_0^\infty u^{\frac{5}{2}} e^{-u}\,\mathrm{d}u$$

Recall the Gamma Function:

$$\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\,\mathrm{d}u$$

So, if we manipulate the exponent for u, we can get it into the right form and condense the expression into Gamma form ($\Gamma(a)$).

$$\int_0^\infty x^6 e^{-4x^2}\,\mathrm{d}x = \frac{1}{256}\int_0^\infty u^{\frac{7}{2}-1} e^{-u}\,\mathrm{d}u = \frac{1}{256}\cdot\Gamma\left(\frac{7}{2}\right)$$

Now, we have seen something similar with fractions, like in Problem 1. Let's use the recurrence relation to break it down.

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \cdots = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

But, we know that $\Gamma\frac{1}{2} = \sqrt{\pi}$, so plugging everything in now,

$$\int_0^\infty x^6 e^{-4x^2}\,\mathrm{d}x = \frac{1}{256}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

Finally, that gives you:

$$\boxed{\int_0^\infty x^6 e^{-4x^2}\,\mathrm{d}x = \frac{15\sqrt{\pi}}{2048}}$$

First, we'll pull the x out to simplify the radical. Also use your log rules to simplify the argument of the natural log.

5 $$\int_0^1 x^3\sqrt{x\ln\left(\frac{1}{x}\right)}\,\mathrm{d}x = \int_0^1 x^3\sqrt{x}\sqrt{-\ln(x)}\,\mathrm{d}x = \int_0^1 x^{\frac{7}{2}}\sqrt{\ln-(x)}\,\mathrm{d}x$$

Let's do a u-sub to start,

$$\text{Let: } u = -\ln(x) \implies x = e^{-u}, \qquad \mathrm{d}x = -e^{-u}\,\mathrm{d}u$$

Adjust the bounds,

$$\text{When: } x = 0 \implies u = \infty, \qquad x = 1 \implies u = 0$$

Substitute in all the values,

$$\int_0^1 x^3\sqrt{x\ln\left(\frac{1}{x}\right)}\,\mathrm{d}x = \int_\infty^0 e^{-\frac{7}{2}u}\sqrt{u}\left(-e^{-u}\right)\,\mathrm{d}u$$

Flip the bounds and let the negatives cancel,

$$\int_0^1 x^3\sqrt{x\ln\left(\frac{1}{x}\right)}\,\mathrm{d}x = \int_0^\infty e^{-\frac{9}{2}u}u^{\frac{1}{2}}\,\mathrm{d}u$$

We're going to do another substitution, because we need the exponent on the e to be $-u$.

$$\text{Let: } t = \frac{9}{2}u \implies u = \frac{2}{9}t, \qquad \mathrm{d}t = \frac{9}{2}\,\mathrm{d}u$$

Now, substitute it all in, simplify the constants and factor them out.

$$\int_0^1 x^3\sqrt{x\ln\left(\frac{1}{x}\right)}\,\mathrm{d}x = \frac{2}{9}\int_0^\infty e^{-t}\left(\frac{2}{9}t\right)^{\frac{1}{2}}\,\mathrm{d}t = \frac{2\sqrt{2}}{27}\int_0^\infty e^{-t}t^{\frac{1}{2}}\,\mathrm{d}t$$

Recognize the Gamma Function, and use recurrence relation,

$$\int_0^1 x^3\sqrt{x\ln\left(\frac{1}{x}\right)}\,\mathrm{d}x = \frac{2\sqrt{2}}{27}\int_0^\infty e^{-t}t^{\frac{3}{2}-1}\,\mathrm{d}t = \frac{2\sqrt{2}}{27}\Gamma\left(\frac{3}{2}\right) = \frac{2\sqrt{2}}{27}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

Finally, cancel the 2's and you're finished!

$$\boxed{\int_0^1 x^3\sqrt{x\ln\left(\frac{1}{x}\right)}\,\mathrm{d}x = \frac{\sqrt{2\pi}}{27}}$$