King's Rule

By SMC Integration Club

King's Rule specifically states:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x \tag{1}$$

Essentially, it says you can reflect a function horizontally about the midpoint of your integration interval without changing the actual integral. Let's go over how that works.

We're going to start with a substitution:

$$I = \int_a^b f(x)\,\mathrm{d}x, \qquad \text{let } x = a+b-t, \text{ then } \mathrm{d}x = -\,\mathrm{d}t$$

Now, you need to change the integration bounds because of the substitution,

$$\text{When } x = a, \qquad a = a+b-t \quad \Longrightarrow \quad t = b$$

$$\text{When } x = b, \qquad b = a+b-t \quad \Longrightarrow \quad t = a$$

As you can see, the bounds reverse.
Therefore,

$$I = \int_b^a f(a+b-t)(-\,\mathrm{d}t) = -\int_b^a f(a+b-t)(\,\mathrm{d}t)$$

Now, we can swap the bounds and absorb the negative,

$$-\int_b^a g(t)\,\mathrm{d}t = \int_a^b g(t)\,\mathrm{d}t$$

So,

$$I = \int_a^b f(a+b-t)\,\mathrm{d}t$$

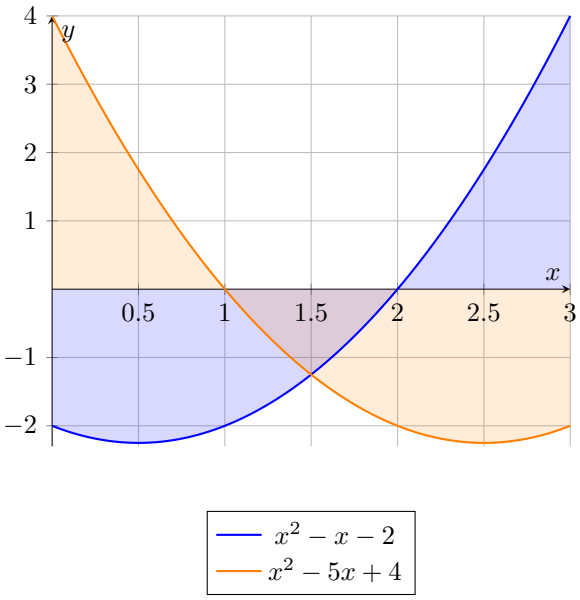
Lastly, since t was just a dummy variable, we can rename it to x.

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x$$

We'll use the following integral as an example:

$$\text{Original Function: } \int_0^3 (x^2 - x - 2)\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_0^3 = -\frac{3}{2}$$

$$\text{King's Rule Alteration: } \int_0^3 (x^2 - 5x + 4)\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_0^3 = -\frac{3}{2}$$



Reminder of some trigonometric facts:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \qquad \sin(-x) = -\sin(x)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x) \qquad \cos(-x) = \cos(x)$$

Let's try some problems now:

1) $$\int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\sin(x)}}{\sqrt[3]{\sin(x)} + \sqrt[3]{\cos(x)}}\,\mathrm{d}x$$

2) $\displaystyle\int_0^{\infty} \frac{1}{x^8 + x^6 + x^2 + 1}\,\mathrm{d}x$

3) $\displaystyle\int_1^4 \frac{\ln(x+1)}{\ln(x+1)+\ln(6-x)}\,\mathrm{d}x$

4) $\displaystyle\int_0^5 \frac{1+\ln\left(\frac{x}{5-x}\right)}{2x^2-10x+25}\,\mathrm{d}x$

5) $\displaystyle\int_0^1 \frac{x^3\left(e^x + e^{1-x}\right)}{3x^2 - 3x + 1}\,\mathrm{d}x$